Although $\delta \boldsymbol{R} \boldsymbol{T}_{i,j} \delta \boldsymbol{R}^T$ could be full rank, multiplication by a rank-deficient $\boldsymbol{B}$ in Eq. (26) results in a loss of information, and Eq. (23) no longer accurately approximates the attitude error covariance. In practical terms for this example, one or more diagonal terms in $\boldsymbol{P}_\Theta$ will appear as zero when in fact correlations do exist. This argument can be extrapolated to realize that, as the condition number of either $\boldsymbol{B}$ or $\boldsymbol{S}$ degenerates, the fidelity of $\boldsymbol{P}_\Theta$ as a true representation of attitude error uncertainty decreases.

Conclusions

The determination of attitude from differential GPS range measurements was posed as an orthogonal Procrustes problem so that a closed-form solution could be found. It was shown that the attitude determination problem could be solved in not one but two different methods. The first method allowed for coplanar baselines, whereas the second method allowed for planar satellite geometry. Subsequent sensitivity and covariance analysis showed that both solutions yielded unbiased estimates. For small attitude changes, a general expression was derived for the covariance of the attitude solution that showed how poor geometry could adversely affect solution quality.

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New Proportional Navigation Law for Ground-to-Air Systems

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I. Introduction

GUIDANCE applications in ground-to-air systems are characterized by high requirements in terms of small miss distances against fast moving targets. Modern defense systems are typically equipped with highly sophisticated subsystems, both onboard the intercepting missiles and as part of the supporting systems, for example, radar systems, enabling good estimates of the interception time to go. Guidance methods that exploit such estimates are more likely to become candidates for realistic applications, whereas simple proportional-navigation (PN) guidance will usually fall short with respect to satisfying the tough requirements.

It is well known that PN (with $N' = 3$) is, in fact, an optimal strategy for the linearized problem,[1,2] when the cost J is the control effort, as follows:

$$J = \int_0^{t_f} u^2(t)\,\mathrm{d}t \tag{1}$$

where u is the missile's lateral acceleration and t_f is the collision time (the elapsed time from the beginning of the end game till interception).

Improved guidance schemes can be obtained with an appropriate selection of a cost function that replaces J (see Ref. 2). This paper employs an exponential term in J. (See Refs. 3 and 4 for exponentially weighted quadratic performance index.)

We let

$$J = \int_0^{t_f} e^{-k(t_f - t)} u^2(t)\,\mathrm{d}t \tag{2}$$

The motivation for this cost is for guiding aerodynamically maneuvering ground-to-air missiles that loose their maneuverability as the air density decreases. Because the air density can be approximated by an exponential term (as a function of altitude), it makes sense to weigh maneuvers in this manner, whereby the penalty for late maneuvers is higher than for earlier ones. As will be shown, this cost leads to a new PN law with a time-varying navigation gain.

The paper is organized as follows: In the next section the standard two-dimensional problem geometry and the mathematical modeling will be reviewed. The problem formulation and analysis by the optimal control theory will be presented in Sec. III. In Sec. IV some numerical results are presented, and in Sec. V the Note is summarized.

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II. Mathematical Modeling

We shall make the following assumptions:

1) The end game is two dimensional and gravity is compensated independently.

2) The speeds of the pursuer (the missile) P and the evader (the target) E are constant during the end game (approximately true for short end games).

3) The trajectories of P and E can be linearized around their collision course.

4) The pursuer can estimate of the time to go and can measure the line-of-sight rate.

We assume that the collision condition is satisfied (Fig. 1), namely,

$$V_p \sin(\gamma_{p0}) - V_e \sin(\gamma_{e0}) = 0 \tag{3}$$

where V_e and V_p are the pursuer's and evader's velocities and γ_{p0} and γ_{e0} the pursuer's and evader's nominal heading angles, respectively.

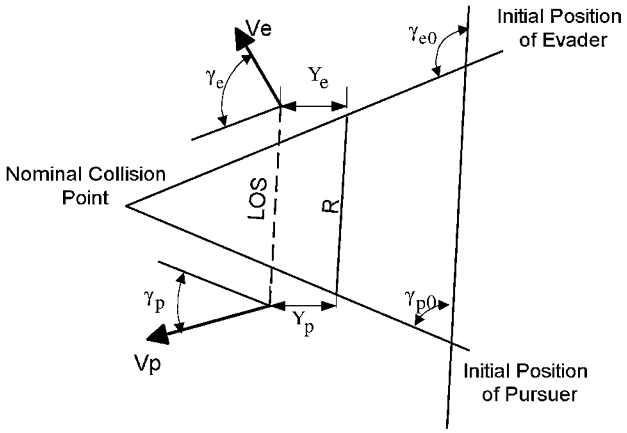

Fig. 1 Problem geometry.

In this case, the nominal closing velocity V_c is given by

$$V_c = -\dot{R} = V_p \cos(\gamma_{p0}) - V_e \cos(\gamma_{e0}) \tag{4}$$

and the (nominal) terminal time is given by

$$t_f = R/V_c \tag{5}$$

where R is the nominal length of the line of sight.

If we allow Y_e and Y_p to be the separation (Fig. 1) of the pursuer and the evader, respectively, from the nominal line of sight, and let y be the relative separation, namely, $y \equiv Y_e - Y_p$, we obtain the following dynamic equation:

$$\dot{y} = \dot{Y}_e - \dot{Y}_p = V_e \sin(\gamma_{e0} + \gamma_e) - V_p \sin(\gamma_{p0} + \gamma_p) \tag{6}$$

where γ_p and γ_e are the deviations from the baseline of the pursuer's and evader's headings, respectively, as a result of control actions applied. If these deviations are small enough, we may use small angles approximation to obtain

$$\sin(\gamma_{p0} + \gamma_p) \approx \sin(\gamma_{p0}) + \cos(\gamma_{p0})\gamma_p \tag{7}$$

$$\sin(\gamma_{e0} + \gamma_e) \approx \sin(\gamma_{e0}) + \cos(\gamma_{e0})\gamma_e \tag{8}$$

Substituting the results into Eq. (9), we find that $\dot{y}$ becomes

$$\dot{y} = \dot{Y}_e - \dot{Y}_p = V_e \cos(\gamma_{e0})\gamma_e - V_p \cos(\gamma_{p0})\gamma_p \tag{9}$$

We can also find an expression for the line-of-sight (LOS) angle and its rate of change. Recall that λ is the LOS angle, and without loss of generality, let $\lambda(t_0) = 0$. We observe that $\lambda(t)$ is

$$\lambda(t) = y/R \tag{10}$$

hence,

$$\dot{\lambda}(t) = \frac{d}{dt}\left(\frac{y}{R}\right) = \frac{y}{V_c(t_f - t)^2} + \frac{\dot{y}}{V_c(t_f - t)} \tag{11}$$

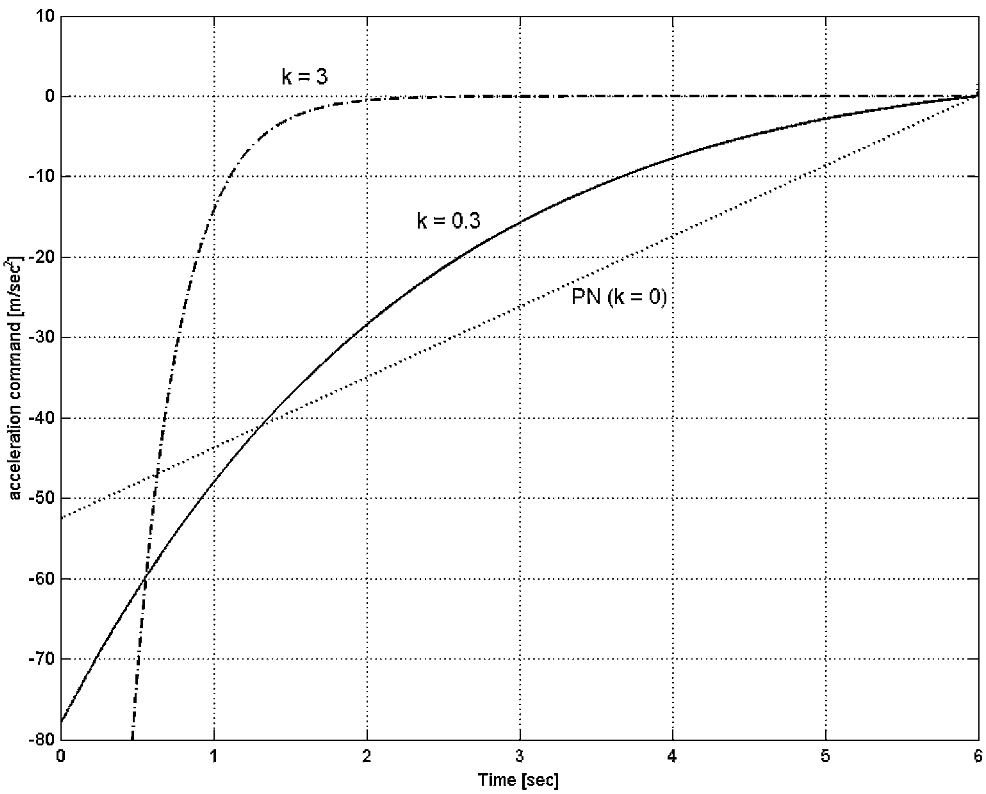

Fig. 2 Missile acceleration due to heading error.

III. Problem Formulation and Analysis

Define

$$x_1 = y, \qquad x_2 = \frac{dy}{dt}, \qquad x_3 = -V_p \cos(\gamma_{p0})\frac{d\gamma_p}{dt} \tag{12}$$

$$u = -V_p \cos(\gamma_{p0})\frac{d\gamma_{p_c}}{dt}, \qquad w = V_e \cos(\gamma_{e0})\frac{d\gamma_e}{dt} \tag{13}$$

We assume now a nonmanevering target, that is, $w = 0$. Dealing with manuvering targets can be pursued along the same lines. [See Ref. 2 for a similar treatment with J as in Eq. (1).] We consider the following minimization problem related to the ideal dynamics guidance problem:

$$\min_u J \qquad \text{subject to} \qquad \dot{x} = Ax + Bu \tag{14}$$

where

$$J = \frac{b}{2}x_1^2(t_f) + \frac{c}{2}x_2^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f} e^{-k(t_f - t)}u^2(t)\,dt$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{15}$$

First-order necessary conditions for optimality involve the following adjoint equations[2]:

$$\dot{\lambda}_1 = 0, \qquad \dot{\lambda}_2 = -\lambda_1 \tag{16}$$

The terminal conditions are

$$\lambda_1(t_f) = bx_1(t_f), \qquad \lambda_2(t_f) = cx_2(t_f) \tag{17}$$

Solving for λ we get

$$\lambda_1 = bx_1(t_f), \qquad \lambda_2 = bx_1(t_f)(t_f - t) + cx_2(t_f) \tag{18}$$

The optimal control satisfies

$$u(t) = -e^{k(t_f - t)}\lambda_2 = -e^{k(t_f - t)}[bx_1(t_f)(t_f - t) + cx_2(t_f)] \tag{19}$$

Substituting $u(t)$ into Eq. (14) and integrating from $t_0 = 0$ to t_f, we get explicit expressions for $x_1(t)$ and $x_2(t)$. By evaluating these expressions at $t = t_f$, we can solve the resulting algebraic equations for the unknown terminal values $x_1(t_f)$ and $x_2(t_f)$. We obtain the explicit expression for $u(t)$:

$$u(t) = -[g_1x_1(t) + g_2x_2(t)] \tag{20}$$

where

$$g_1 = \frac{k\eta\left(bk^3T_{go} - bck^2T_{go} + bck\eta - bck\right)}{-2bk - ck^3 + bc - 2bk^2T_{go}\eta - bck^2T_{go}^2\eta + k^4 + bk^3T_{go}^2\eta + ck^3\eta - 2bc\eta + bc\eta^2 + 2bk\eta} \tag{21}$$

$$g_2 = \frac{k\eta\left(bk^3T_{go}^2 - bck^2T_{go}^2 - 2bckT_{go} + 2bc\eta - 2bc + ck^3\right)}{-2bk - ck^3 + bc - 2bk^2T_{go}\eta - bck^2T_{go}^2\eta + k^4 + bk^3T_{go}^2\eta + ck^3\eta - 2bc\eta + bc\eta^2 + 2bk\eta} \tag{22}$$

$$T_{go} = t_f - t, \qquad \eta = e^{kT_{go}} \tag{23}$$

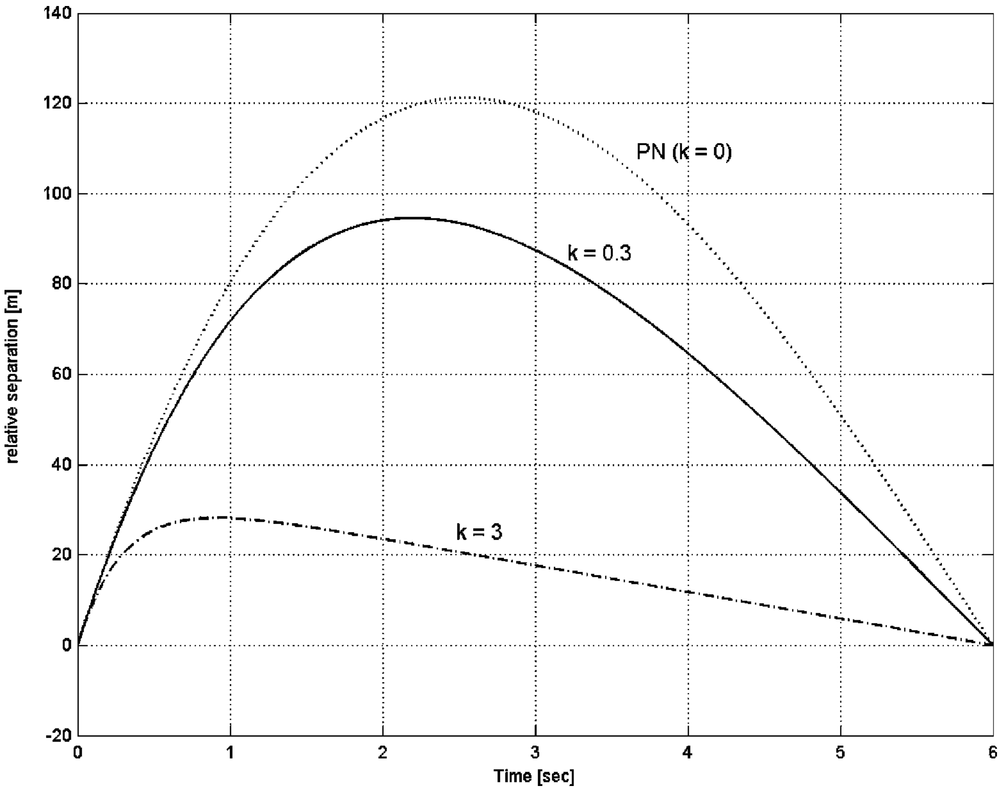

Fig. 3 Relative separation due to heading error.

In the special case $c = 0$ and $b \to \infty$ (pure intercept), we get

$$g_1 = \frac{k^3 T_{\text{go}}}{2 + k^2 T_{\text{go}}^2 - 2k T_{\text{go}} - 2\eta^{-1}} \tag{24}$$

$$g_2 = \frac{k^3 T_{\text{go}}^2}{2 + k^2 T_{\text{go}}^2 - 2k T_{\text{go}} - 2\eta^{-1}} \tag{25}$$

Using Eq. (11)

$$\dot{\lambda} = \frac{x_1 + x_2 T_{\text{go}}}{V_c T_{\text{go}}^2} \tag{26}$$

we obtain a new PN law (in the sequel referred to as a modified PN),

$$u = -N' V_c \dot{\lambda} \tag{27}$$

where

$$N' = \frac{k^3 T_{\text{go}}^3}{k^2 T_{\text{go}}^2 - 2k T_{\text{go}} + 2 - 2\eta^{-1}} \tag{28}$$

Two interesting properties of the new guidance law are

$$\lim_{T_{\text{go}} \to 0} N' = 3 \tag{29}$$

$$\lim_{k \to 0} N' = 3 \tag{30}$$

Note that $N' = 3$ matches the well-known PN law.

IV. Simulation Results

In this section, we consider a numerical example that illustrates the merits of the new guidance law. We analyze the effect on the trajectory of a 2-deg heading error. The end game is assumed to take 6 s, and the conflict is assumed to be a head on with a closing velocity of 3000 m/s. The effect of three guidance laws will be analyzed: 1) PN with $N' = 3$, 2) modified PN with $k = 0.3\ \text{s}^{-1}$, and 3) modified PN with $k = 3\ \text{s}^{-1}$.

Figure 2 shows the missile acceleration for the three different cases. The linear behavior with time of the acceleration under PN guidance is a well-known fact. Notice how the modified PN redistributes the acceleration along the flight. This is done by using higher PN gains at the earlier stages of the conflict as dictated by Eq. (28). The higher is k, the shorter is the accelerating time segment. This affects the trajectory as shown in Fig. 3.

V. Conclusions

A new PN guidance scheme was obtained with an exponentially weighted quadratic cost function. This modified PN, based on time-to-go estimates, redistributes the acceleration along the trajectory. The advantages of the new scheme become clear if we consider the fast reduction in maneuverability for aerodynamically maneuvering ground-to-air missiles. The derivation also provides the possibility to include weights on the terminal relative velocity in the cost function, thus producing more guidance schemes that are worth studying.

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Trigonometric Models for Large-Angle Aerodynamic Force Coefficients

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Introduction

NONTABULAR models of aerodynamic force coefficients used in mass-point characterizations of flight-vehicle motion are helpful for derivation and implementation of optimal guidance laws and predictions of vehicle performance. As interest develops in vehicle designs and tactics for large-α, large-β maneuvers, there is a corresponding increase in the need for aeroforce coefficient models valid in large-angle regimes. The force coefficients may be modeled in many ways to represent data from computational fluid dynamics work, wind-tunnel experiments, and/or flight sensors. Coefficient models may be tabular, analytic, or a blend of the two. By judicious use of analytic representations, the tabular storage of coefficients for large-angle dynamics can be greatly simplified without compromising accuracy, and the analyst can gain greater insight regarding force dependencies on the aeroangles (α and β).

Traditional aeroforce coefficient models are linear and quadratic relationships that are applicable over relatively limited angle ranges unless the parameters within these relationships are themselves varied with the aeroangles. The purpose of this Engineering Note is to illustrate simple steps whereby the aeroangle dependencies may be expressed analytically to make them explicit for both small and large angles. These steps can facilitate analytical treatments and lessen the complexity of stored tables or polynomial representations. The goal is for the parameters in the coefficient models to exhibit dependence only on nonangular variables such as Mach number, altitude, etc.

Classical models for aeroforce coefficients, written in terms of the American National Standards Institute/AIAA air-path (wind-axis) coordinate system are[1,2]

$$C_{D_W} = C_{D_0} + C_{L_W}^2 \big/ [\pi e (\mathit{AR})] \tag{1}$$

$$C_{Y_W} = C_{Y_\beta} \beta \quad \left(\text{nominally } C_{Y_\beta} < 0\right) \tag{2}$$

$$C_{L_W} = C_{L_\alpha} \alpha \tag{3}$$

where α is the angle of attack, defined as $\alpha = \tan^{-1}(w/u)$, and β is the angle of sideslip, defined as $\beta = \sin^{-1}(v/V)$, where u, v, and w are the linear velocity components along the x, y, and z body axes of the vehicle, respectively, whereas V is the resultant of the velocity components. C_{D_W} is the wind-axis drag coefficient. C_{D_0} is the sum of pressure and skin-friction drag coefficients; it is often referred to as the zero-lift drag coefficient. C_{Y_W} and C_{L_W} are the side-force and lift coefficients, respectively. (AR) is the wing planform aspect ratio, and e is the span efficiency factor. C_{Y_β} and C_{L_α} are the side-force derivative and lift-curve slope, respectively. Commonly, C_{D_0}, e, C_{Y_β}, and C_{L_α} are stored as tabular functions of α, β, Mach number, and altitude. The aerocoefficients may be defined for trimmed flight conditions, but often the control-surface deflections are treated as further independent variables.

A number of candidate analytic forms are available for representing aeroforce coefficients; these include regression curve fits using polynomial, exponential, or spline basis functions. Simple

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